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An Extension of Montessus de Ballore's Theorem on the Convergence of Interpolating Rational Functions

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THIS PAPER IS DEDICATED TO PROFESSER J. L. WALSH ON THE OCCASION OF HIS 75TH BIRTHDAY, IN RECOGNITION OF HIS OUTSTANDING CONTRIBUTIONS TO THE THEORIES OF APPROXIMATION, CONFORMAL MAPPING, AND CRITICAL POINTS.

A rational function $r_{\mu\nu}(z)$ is said to be *of type* (μ, ν) if it is of the form

$$r_{\mu\nu}(z) = p_\mu(z)/q_\nu(z), \qquad q_\nu(z) \not\equiv 0,$$

where $p_\mu(z)$ is a polynomial of degree at most μ and $q_\nu(z)$ is a polynomial of degree at most ν. To each function $f(z)$, analytic at $z = 0$, there corresponds a doubly-infinite array known as the Padé table [2, Section 73] whose entries are rational functions $R_{\mu\nu}(z)$ which interpolate to $f(z)$ in the origin. For each pair (μ, ν), the rational function $R_{\mu\nu}(z)$, of type (μ, ν), is chosen so that $f(z) - R_{\mu\nu}(z)$ has a zero of the highest possible order at $z = 0$. Concerning the convergence of these Padé rational functions we have the following important result of R. de Montessus de Ballore [1]:

THEOREM 1. *Let $f(z)$ be analytic at $z = 0$ and meromorphic with precisely ν poles (multiplicity counted) in the disk $|z| < \tau$. Let D denote the domain obtained from $|z| < \tau$ by deleting the ν poles of $f(z)$. Then, for all n sufficiently large, there exists a unique rational function $R_{n\nu}(z)$, of type (n, ν), which interpolates to $f(z)$ in the point $z = 0$ considered of multiplicity $n + \nu + 1$. Each $R_{n\nu}(z)$ has precisely ν finite poles and, as $n \to \infty$, these poles approach, respectively, the ν poles of $f(z)$ in $|z| < \tau$. The sequence $R_{n\nu}(z)$ converges to $f(z)$ throughout D, uniformly on any compact subset of D.*

To prove Theorem 1, Montessus de Ballore used Hadamard's classical results on the location of the polar singularities of a function represented by

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a Taylor series. In the present paper, using only elementary methods from the theory of interpolation, we prove the following generalization of Theorem 1 (cf. [3, Theorem 3]):

THEOREM 2. *Let E be a closed bounded point set whose complement K (with respect to the extended plane) is connected and regular in the sense that K possesses a Green's function $G(z)$ with pole at infinity. Let Γ_σ ($\sigma > 1$) denote generically the locus $G(z) = \log \sigma$, and denote by E_σ the interior of Γ_σ. Let the points*

$$\begin{gathered} \beta_1^{(0)}, \\ \beta_1^{(1)}, \beta_2^{(1)}, \\ \cdots \\ \beta_1^{(n)}, \beta_2^{(n)}, \ldots, \beta_{n+1}^{(n)}, \\ \cdots \end{gathered} \tag{1}$$

(which may not all be distinct) have no limit point exterior to E and satisfy the relation

$$\lim_{n\to\infty} \left| \prod_{i=1}^{n+1} (z - \beta_i^{(n)}) \right|^{1/n} = \Delta \exp G(z), \tag{2}$$

uniformly in z on each closed bounded subset of K, where Δ is the transfinite diameter [4, §4.4] *of E.*

Suppose that the function $f(z)$ is analytic on E and meromorphic with precisely ν poles in E_ρ ($\rho > 1$). Let D_ρ denote the region obtained from E_ρ by deleting the ν poles of $f(z)$. Then for all n sufficiently large there exists a unique rational function $r_{n\nu}(z)$, of type (n, ν), which interpolates to $f(z)$ in the points $\beta_1^{(n+\nu)}$, $\beta_2^{(n+\nu)}, \ldots, \beta_{n+\nu+1}^{(n+\nu)}$. Each $r_{n\nu}(z)$ has precisely ν finite poles and, as $n \to \infty$, these poles approach, respectively, the ν poles of $f(z)$ in E_ρ. The sequence $r_{n\nu}(z)$ converges to $f(z)$ throughout D_ρ, uniformly on any compact subset of D_ρ.

If $f(z)$ is defined as in Theorem 1, we can choose $\rho > 1$ so large that $f(z)$ is analytic on $E : |z| \leqslant \tau/\rho$. Then, by taking all the points (1) to be zero, we deduce Theorem 1 as a special case of Theorem 2.

Proof of Theorem 2. Let $\alpha_1, \alpha_2, \ldots, \alpha_\nu$ be the ν poles of $f(z)$ in E_ρ and set

$$Q_0(z) = 1, \qquad Q_k(z) = \prod_{i=1}^{k} (z - \alpha_i), \qquad 1 \leqslant k \leqslant \nu.$$

Put

$$q_n(z) = \sum_{k=1}^{\nu} a_k^{(n)} Q_{k-1}(z) + Q_\nu(z),$$

and let $\pi_n(z)$ be the unique polynomial of degree at most $n + \nu$ which interpolates to the analytic function $q_n(z)\, Q_\nu(z) f(z)$ in the points $\beta_1^{(n+\nu)},..., \beta_{n+\nu+1}^{(n+\nu)}$. We shall choose the coefficients $a_k^{(n)}$, $1 \leqslant k \leqslant \nu$, so that the polynomial $Q_\nu(z)$ is a factor of $\pi_n(z)$. To show that this is indeed possible, let R, $1 < R < \rho$, be such that the poles $\alpha_1 ,..., \alpha_\nu$ all lie in E_R. For n sufficiently large, we have by the Hermite formula [4, p. 50]

$$\pi_n(z) = \frac{1}{2\pi i} \int_{\Gamma_R} \left[1 - \frac{\omega_{n+\nu}(z)}{\omega_{n+\nu}(t)}\right] \frac{q_n(t)\, Q_\nu(t) f(t)}{t - z}\, dt,$$

where $\omega_{n+\nu}(z) = \prod_{i=1}^{n+\nu+1} (z - \beta_i^{(n+\nu)})$.

Suppose, first, that the α_j are all distinct, i.e., $f(z)$ has only simple poles in E_ρ. Then $Q_\nu(z)$ is a factor of $\pi_n(z)$ if and only if

$$\sum_{k=1}^{\nu} c_{jk}^{(n)} a_k^{(n)} = d_j^{(n)}, \qquad j = 1, 2,..., \nu, \tag{3}$$

where

$$c_{jk}^{(n)} = \frac{1}{2\pi i} \int_{\Gamma_R} \left[1 - \frac{\omega_{n+\nu}(\alpha_j)}{\omega_{n+\nu}(t)}\right] \frac{Q_{k-1}(t)\, Q_\nu(t) f(t)}{t - \alpha_j}\, dt,$$

$$d_j^{(n)} = \frac{-1}{2\pi i} \int_{\Gamma_R} \left[1 - \frac{\omega_{n+\nu}(\alpha_j)}{\omega_{n+\nu}(t)}\right] \frac{Q_\nu(t)^2 f(t)}{t - \alpha_j}\, dt.$$

From (2) we deduce that

$$\lim_{n\to\infty} c_{jk}^{(n)} = \frac{1}{2\pi i} \int_{\Gamma_R} \frac{Q_{k-1}(t)\, Q_\nu(t) f(t)}{t - \alpha_j}\, dt, \qquad 1 \leqslant j, k \leqslant \nu,$$

and, by Cauchy's integral theorem, we have

$$\int_{\Gamma_R} \frac{Q_{k-1}(t)\, Q_\nu(t) f(t)}{t - \alpha_j}\, dt \begin{array}{ll} = 0, & \text{for } k > j, \\ \neq 0, & \text{for } k = j. \end{array}$$

Hence,

$$\lim_{n\to\infty} \det[c_{jk}^{(n)}] = \prod_{l=1}^{\nu} \frac{1}{2\pi i} \int_{\Gamma_R} \frac{Q_{l-1}(t)\, Q_\nu(t) f(t)}{t - \alpha_l}\, dt \neq 0, \tag{4}$$

and, so, for n sufficiently large, the linear system (3) can be solved uniquely for the coefficients $a_k^{(n)}$. Furthermore, since $d_j^{(n)} \to 0$ as $n \to \infty$, it follows from (4) and Cramer's rule that, for each k, $1 \leqslant k \leqslant \nu$, we have $a_k^{(n)} \to 0$ as $n \to \infty$.

Thus

$$\lim_{n\to\infty} q_n(z) = Q_\nu(z), \tag{5}$$

uniformly on each bounded subset of the plane.

We consider now the case where $f(z)$ has at least one multiple pole in E_ρ. Suppose, for the sake of definiteness, that α_1 is a pole of order $\mu(\geqslant 2)$ of $f(z)$, i.e., $\alpha_1 = \alpha_2 = \cdots = \alpha_\mu$. The polynomial $Q_\nu(z)$ is a factor of $\pi_n(z)$ if and only if the coefficients $a_k^{(n)}$ satisfy the linear system

$$\sum_{k=1}^{\nu} C_{jk}^{(n)} a_k^{(n)} = D_j^{(n)}, \qquad j = 1, 2,\ldots, \nu, \tag{6}$$

the first μ equations of which are obtained by setting the derivatives $\pi_n(\alpha_1)$, $\pi_n{}'(\alpha_1),\ldots, \pi_n^{(\mu-1)}(\alpha_1)$ equal to zero. As in the case of simple poles it is easy to show each sequence $\{C_{jk}^{(n)}\}_{n=0}^{\infty}$ has a limit, say C_{jk}, and that these limits satisfy $C_{jj} \neq 0$, $C_{jk} = 0$ for $k > j$. In particular, we have

$$C_{jk} = \frac{(j-1)!}{2\pi i} \int_{\Gamma_R} \frac{Q_{k-1}(t)\, Q_\nu(t) f(t)}{(t-\alpha_1)^j}\, dt, \qquad 1 \leqslant j \leqslant \mu, \quad 1 \leqslant k \leqslant \nu.$$

Thus, $\lim_{n\to\infty} \det [C_{jk}^{(n)}] = \prod_{l=1}^{\nu} C_{ll} \neq 0$, and, so, the system (6) can also be solved uniquely for the $a_k^{(n)}$. Furthermore, (5) remains valid.

Now set $r_{n\nu}(z) \equiv \pi_n(z)/q_n(z)\, Q_\nu(z)$. Then by our choice of the coefficients $a_k^{(n)}$, we have that $r_{n\nu}(z)$ is a rational function of type (n, ν). Moreover, since the points α_j lie exterior to E, it follows from (5) that, for n sufficiently large, $q_n(z)$ is different from zero at each of the points $\beta_1^{(n+\nu)},\ldots, \beta_{n+\nu+1}^{(n+\nu)}$. Hence, $r_{n\nu}(z)$ must interpolate to $f(z)$ in these points.

If $s_{n\nu}(z)$ is another rational function of type (n, ν) which interpolates to $f(z)$ in the points $\beta_1^{(n+\nu)},\ldots, \beta_{n+\nu+1}^{(n+\nu)}$, then the difference $r_{n\nu}(z) - s_{n\nu}(z)$ is a rational function of type $(n+\nu, 2\nu)$ which has at least $n+\nu+1$ zeros. Such a rational function must be identically zero. Thus, $r_{n\nu}(z)$ is uniquely determined by its interpolation property.

Now let $S \subset D_\rho$ be compact, and choose λ, $1 < \lambda < \rho$, so that $S \subset E_\lambda$. Then, for $\lambda < \sigma < \rho$, we have

$$q_n(z)\, Q_\nu(z) f(z) - \pi_n(z) = \frac{1}{2\pi i} \int_{\Gamma_\sigma} \frac{\omega_{n+\nu}(z)\, q_n(t)\, Q_\nu(t) f(t)}{\omega_{n+\nu}(t)(t-z)}\, dt, \qquad z \text{ on } S,$$

and, hence, from (2) and the uniform boundedness of the $q_n(t)$ on Γ_σ, there follows

$$\overline{\lim_{n\to\infty}}\, [\max |\, q_n(z)\, Q_\nu(z) f(z) - \pi_n(z)|;\ z \text{ on } S]^{1/n} \leqslant \lambda/\sigma.$$

But (5) implies that, for n sufficiently large, $|q_n(z)\, Q_\nu(z)|$ is uniformly bounded below by a positive constant for z on S, and so,

$$\overline{\lim_{n\to\infty}}\, [\max |f(z) - r_{n\nu}(z)|;\ z \text{ on } S]^{1/n} \leqslant \lambda/\sigma < 1.$$

Finally, note that $r_{n\nu}(z)$ has ν formal poles, namely, the zeros of $q_n(z)$, and that as $n \to \infty$, these poles approach, respectively, the ν poles of $f(z)$ in E_ρ. However, since

$$\lim_{n\to\infty} \pi_n(z)/Q_\nu(z) = Q_\nu(z)\, f(z),$$

uniformly in z in a neighborhood of each of the points α_j, it follows that, for n sufficiently large, no zero of the polynomial $\pi_n(z)/Q_\nu(z)$ can be a zero of $q_n(z)$. Thus, the ν formal poles of $r_{n\nu}(z)$ are *actual* poles. This completes the proof of Theorem 2.

An easy consequence of the above proof is the following

COROLLARY. *The rational functions $r_{n\nu}(z)$ of Theorem 2 satisfy*

$$\overline{\lim_{n\to\infty}}\, [\max |f(z) - r_{n\nu}(z)|;\ z \text{ on } E]^{1/n} \leqslant 1/\rho.$$

REFERENCES

1. R. DE MONTESSUS DE BALLORE, Sur les fractions continues algébriques, *Bull. Soc. Math. France*, **30** (1902), 28–36.
2. O. PERRON, "Die Lehre von den Kettenbrüchen," 2nd ed., Chelsea, New York, 1929.
3. J. L. WALSH, The convergence of sequences of rational functions of best approximation with some free poles, *in* "Approximation of Functions" (H. L. Garabedian, Ed.), Elsevier, Amsterdam, 1965.
4. J. L. WALSH, Interpolation and approximation by rational functions in the complex domain, 5th ed., Colloq. Publ. Vol. 20, American Mathematical Society, Providence, RI 1969.